Linguistic competence and mathematics learning: the tools of e-learning

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This paper addresses the issue of mathematics education in e-learning platforms. In the frame of discursive approach to mathematics learning, the discussion is focused on multisemioticity and multivariety, as they characterize mathematical practice and students' linguistic competence seems to be strictly linked to their success in mathematics learning. E-learning platforms offer plenty of opportunities to plan and implement activities apt to improve such competence. To this aim, the tools available are investigated and some examples of their use are shown.

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1 Introduction

The main question which is the thread of this paper is one which rises spontaneously even in people not interested in education: *why does mathematics seem so very difficult to learn and why is this learning so prone to failure?* (Sfard, 2001, p. 15). Strictly linked to this one, other two questions rise, which consider the nature of mathematics and its difference from other scientific fields: *What is the nature of the difficulties, frequently insurmountable, that many students have with comprehension of mathematics? What characterizes mathematical activity from a cognitive point of view?* (Duval, 2006, p. 1-2).

In the following, we look at a specific theoretical framework, the so called *discursive approach* to mathematics learning, in order to give an answer to the previous questions. In this frame, we investigate features of mathematics distinguishing it from other scientific fields, in particular we discuss multisemioticity and multivariety and the strict link among students' linguistic capabilities and their learning success. Therefore it comes out the need of planning and implementing educational activities in order to improve such competencies in students.

E-learning platforms offer plenty of opportunities to make actual this aim. In a further section we analyze the tools available and how to use them in activities devoted to foster students' progress from the point of view of multisemioticity and multivariety of the language of mathematics.

Finally we briefly discuss possible future trends.

2 Learning and teaching mathematics: reference theoretical framework

The answer the above questions first of all requires to define what is learning. Numerous scientific theories on cognitive functioning regarding learning as information storing according to mental representations, have brought researchers in mathematics education to adopt at first the metaphor of learning as acquisition of knowledge, where the word 'acquisition' points out the individual character of the effort, and the acquisition can occur as passive reception or active construction, leading to a personalized version of concepts and procedures (Sfard, *op. cit.*, pp. 20-21). Not always the personalized version of the concept matched the 'official', academic, scientific version, and so the active construction has often lead to generate what are named *misconceptions, images or tacit models*. In order to contrast such phenomenon, various models have been proposed, for instance the a-didactical situations theory (Brousseau, 1997), which foresees a continuous interaction with the situation until the student has reached the knowledge – hidden goal – giving rise to an *institutionalization* phase, aimed to 'certificate' that the acquired knowledge is not 'misconception'.

In this framework, where knowledge is an 'object to acquire', learning-with-understanding, defined by the cognitive psychologists as the ability to make links new knowledge to what already possessed, becomes the 'process of acquisition', taking for granted its use in other situations whenever appropriate (Sfard, *op. cit.*, p. 21). Essentially learning by acquisition keeps the cognitive activities apart from their context.

This viewpoint seems to be restrictive in order to explain that comprehension which often underpins choices and decisions taken by an individual. For such reason, the metaphor of learning by acquisition has been extended by Sfard through the communicational metaphor, whose basic principle consists in *conceptualizing thinking as a case of communication, that is thinking is nothing but our communicating with ourselves, not necessarily inner, and not necessarily verbal* (Sfard, *op. cit.*, p. 26).

In this new perspective, since communication can be defined as the tentative of a person to make an interlocutory act and to think or feel according her intentions, the phenomena of mutual and self regulations assume particular relevance. Looking at cognition as communications, it follows that thinking is subordinated to, and informed by, the necessity of making communication effective (Sfard, *op. cit.*, p. 27). The key of communication is the *discourse* which is defined as any specific instance of communication, both diachronic and synchronic, with ourselves and with others, mainly verbal and with the help of other semiotic systems.

Learning mathematics may now be defined as an initiation to mathematical discourse, that is, initiation to a special form of communication known as mathematical (Sfard, *op. cit.*, p.28). To this aim two key factors will be considered in studying thinking as communication: the *mediation tools*, that are the languages which are of use for communicating, and the *meta-discursive rules*, which regulate the communicative effort.

On the contrary of what usually 'tool' means, Sfard considers languages not on as couriers of pre-existing meanings, but as constructors of the meanings themselves. From this viewpoint, the language hardly influences thinking. Differing from other fields, such as zoology or chemical, which can be defined as discourses about the animals or the chemical matters, in the case of mathematics, we cannot distinguish the discourse and its objects, because the mathematical objects are themselves discursive constructions and then part of the discourse (Sfard, 2008, p. 161).

In this sense mathematics is an autopoietic system, that is a system which produces what it talks about (Sfard, *op. cit.*, p. 194). The mathematical discourses are characterized by the tools they use (words and visual tools) and by the shape and the results of their processes (routines or approved narrations they produce). If thinking is a type of communication and mathematics education is strictly linked to the participation to a discourse, then the way the representation and communications tools are realized by becomes fundamental. The quality of language influences the quality of thinking and this requires educational attention to the correspondence between semiotic activities and linguistic competency of the participants (Ferrari, 2004b).

In the following sections we try to answer to the initial questions, from the described theoretical perspective and going to analyze two fundamental dimensions of the mathematical discourse.

3 The role of multisemioticity/multivariety in mathematics teaching and learning

3.1 MULTISEMIOTICITY

In a framework where thinking is seen as communication, the languages adopted and their features are remarkably important. This holds a fortiori as far as mathematics is concerned, since the semiotic systems it adopts do not play representative or communicative functions only, but in some cases they are themselves the object of mathematical research (as happens with symbolic notations), in other cases they have highly specific features (such as geometrical figures, which have played a fundamental role in the history of mathematics, or verbal language itself, which will be dealt with in next section).

It is worthwhile to underline that mathematical objects are accessible and treatable as far as they can be represented. Duval (1995) claims that the processes of construction of knowledge and those of representation are closely intertwined: there is no *noesis* without *semiosis*.

Duval distinguishes between two basic transformations regarding semiotic representations:

- Treatment refers to transformations of representations within the same semiotic system. Calculations done within the same number notation system, the resolution of an equation or the decomposition and recomposition of a geometrical shape are examples of treatments.
- Conversion refers to the transformation of a representation in a semiotic system into another semiotic system. The transition from an equation like $y=f(x)$ to a drawing of the graph of the corresponding function, or from a fraction to a decimal number, are examples of conversions.

Treatments are sometimes linked to procedures that depend on the notation system adopted. Conversions allow people to use different semiotic systems in order to address a problem, each one stressing different properties. The opportunity to use more semiotic systems to represent mathematical ideas and procedures provides two main properties: it allows people to (1) distinguish between the sign (and its properties) and the reference (and its properties), and (2) adopt the most effective treatments available.

The distinction between the 'object' represented and its representation is essential: two different representations of a same 'object' do not exactly provide the same information, as happens with the graph of a function compared to its algebraic representation. Any representation highlights some aspects but not all: "*Any representation is cognitively partial compared to what it represents*" (Gagatsis, 2003). In mathematics, contrary to other scientific domains, the need for semiotic representations for accessing and handling abstract objects implies that it is difficult to distinguish between an object and its representation. This in turn implies the impossibility of applying knowledge outside narrow learning contexts, as it does not promote cognitive transfers and further learning gains (*Ibidem*).

It is therefore essential from the point of view of teaching to give the opportunity to access multiple representations of the same 'object' as a necessary condition to learn to distinguish an object from its representations.

Arguments of this type lead Duval to the conclusion that the contemporary mobilization of at least two systems of representation along with the ability to quickly switch from one to another (coordination of semiotic systems) is essential for the understanding of mathematics.

3.2 MULTIVARIETY

Ferrari (2004a) highlights the need for a broad definition of "language of mathematics", whose peculiarity lies not only in symbolic component, but also involves verbal texts (oral and written) and figural representations. Moreover, he claims that in order to explain the functions of mathematical language in educational context it is necessary to take into account both its specificity (the needs for representation and treatment mentioned above) and the fact that it involves people who have to communicate with each other. The verbal component of the language of mathematics is faced with two fundamentally different functions: to represent mathematical knowledge and be able to communicate in the classes and with teachers. The realization, often simultaneously, of these functions, requires, for each of them, the use of appropriate linguistic forms. To make only a quick example, from the point of view of the description of mathematical knowledge, it is correct naming 'isosceles' a triangle with sides two by two congruent, as an equilateral triangle is a special case of an isosceles triangle. From the point of view of communication between people, however, it is much more effective name it 'equilateral', since this choice fulfills to the principles of communicative cooperation (for example, Grice, 1975), while the use of 'isosceles', while mathematically correct, might suggest that the triangle is not equilateral.

For an effective analysis of the language of mathematics, which go beyond the sterile controversies about the degree of 'formalism' of the same, Ferrari (2004a) adopted the perspective of functional linguistics (Halliday, 1974; 1985; 2004; O'Halloran, 2005). If one assumes the perspective of functional linguistics, what distinguishes the mathematical language from everyday language is not so much the degree of "formalism" that they adopt, but the different functions that the two are called upon to play. According to Halliday (see Ferrari, *op. cit.*, p. 35), the functions of languages are: the *ideational*, which concerns the identification of referents and the truth or falsity of the statements, the *interpersonal*, which covers the mutual influences between the participants in the exchange, the *textual* one, which concerns the construction of texts. The latter can not be separated from the *context in*: the interpretation/production of a text (regardless of the sign system used, namely verbal text, oral or written, figure, formula, diagram) students are strongly influenced by their perception of the situation and of the goals of the interaction. What links the text to the context is the *register*, understood as a linguistic variety based on use. Depending on the functions of language, we use different registers. We distinguish *literate* registers evolved from *colloquial* ones; the former are used in communication, in most of the texts of fiction, in the legal context and in many other situations, the latter are used in everyday communication, usually by people who share the context of situation in which the communication takes place. The same person, however, can use one or another register depending on the context, so the distinction is once again functional. Ferrari stresses the need for considering all the registers that are used to make and communicate mathematics, at any level. For mathematical language (Ferrari, *op. cit.*, p 48) we mean a system not only multisemiotic (as seen in the previous section), but also multivariate (which includes a wide range of registers).

Ferrari (*op. cit.*) has shown that the language of mathematics presents in extreme form the features that distinguish the literate registers from the colloquial ones (e.g., close syntactic organization, lesser dependence on the context, hierarchical structure of the texts, lexicalization, nominalization). In other words, the linguistic forms generally used in mathematics, are very far from colloquial uses. Based on this, many of the students' difficulties can be connected to their inability to recognize the two ways of using language (mathematical and colloquial registers) and appropriately switch between them. Familiarity with literate written registers and their use is therefore seen as a favorable starting point for the learning of mathematics, if not a prerequisite.

It is also worthwhile to emphasize that verbal language, unlike other representations, is able to perform the function of driving and controlling thought and operating as a metalanguage. To this end, students' ability to consciously mobilize their own language resources must be promoted in order to achieve different purposes and to control the products, or to manage the use of different registers, among which at least one sufficiently advanced (Ferrari, *op. cit.,* p. 74-75). This also requires the overcoming of teaching practices based on teachers and students' beliefs on the subsidiary function of non-symbolic representations and the clear-cut separation between mathematics and language.

4 What happens in the learning environments

Duval, for the coordination of semiotic systems, and Ferrari, for the linguistic and visual competence, both recognize the need for a teaching action to overcome the difficulties identified. In both cases, the authors stress that the skills required are not "innate" or "spontaneous", but must be "educated". This emphasis is not trivial, since often teachers, because of their beliefs along with the constraints imposed by school schedules, might be induced to take these skills for granted. In this section we will see examples of how technology can help to implement teaching units useful for this purpose.

In e-learning, the discursive approach to mathematics learning, along with the specificity of multisemioticity and multivariety, blends well with the overall picture of Laurillard's Conversational Framework (1993), according to which the process of teaching / learning is modeled through a continuous exchange and interactive relationship between teacher and student and between student and him/herself. The teacher builds an environment the student interacts in/with through a cycle "goal-action-feedback-action modified" and the dialogue between the student and the teacher encourages discursive reflection on the experience to allow him/her to continue in a new learning cycle.

E-learning platforms present a range of opportunities to implement this approach. Modules as 'workshop', 'forum', 'wiki', 'blog', are by their very nature fit for a discursive approach, but also modules as 'task' and 'quiz' can be extremely useful to the issue of multisemioticity and multivariety.

4.1 The quiz module

In the Quiz module each item (stimulus, question, problem,...) can include verbal texts, symbolic expressions, images. The student response often consists of one or more clicks on locations of the screen. In some cases (numerical answer, short answer,...) the student is required to prepare a text, verbal or symbolic.

It is needless to point out the limits of the use of selected-response quiz items (such as multiple choice or true/false) in mathematics education. However, they provide good opportunities, such as easy accessibility by students, the possibility for them to get an immediate feedback and the fact that the activity, once set, can also be used in the absence of a tutor. From the point of view of multisemioticity and coordination of semiotic systems, in order to get the quizzes to be effective tools, it must be clear that the items should not be aimed at the verification of content acquisition, but be focused on competence. This requires an explicit programming that openly includes multisemiotic texts. Each *item* of a quiz may include verbal texts, symbolic expressions, images. It is clear that preparing a content-oriented quiz does not require the same effort that can take an item like the one shown in the following figure:

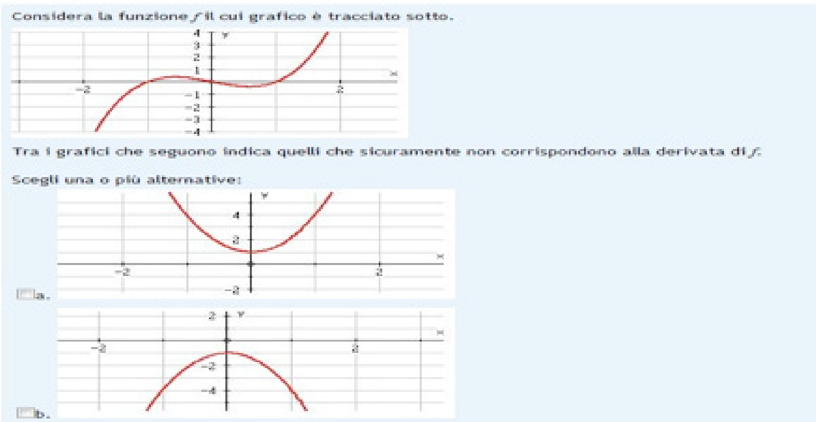

Figure 1

The limits of selected-response items, however, are many and important. A first difficulty is related to the fact that students are not required to produce a text or a representation, whereas the ability at doing so is an essential goal of mathematics education. A second difficulty arises from the fact that students are not required to set up the solution process of a problem or to frame a question, but only to choose from a few options. It is not hard to think of examples of questions or problems whose short-answer and multiple-choice versions require completely different skills, processes and strategies. Having to choose out of 4-5 answers instead of a blank sheet of paper, provide a large amount of information that narrow the field of possibilities and allow inferences that might ignore the knowledge of the subject and depend strongly on the quality of the distractors. The first difficulty is insurmountable, and needs to be addressed by other means, as we shall see below, the second one can be at least partially overcome through a well-targeted selection of items (and the related distractors) that still require a good understanding of concepts involved and inhibit implicatures and any attempts to answer randomly. A choice of distractors that meets the criteria of symmetry, or completeness inevitably provides clues to a cunning solver. Since the use of these tools is primarily educational, the risk is not so much to distort assessment as to let users get distorted information on their preparation. Especially in relation to issues of this type it is appropriate that, in all cases in which it is possible, is present between the options also an entry like 'None of the other answers is adequate', and that this is the response to be chosen in a sufficient number of cases. This does not inhibit the automatic evaluation of the results, as the appropriate response is not required, but it gives information on users' critical skills and discourages improper strategies, those that Vinner (1997) calls 'pseudoanalytic'.

4.2 The 'task'

The module Task acts as a counterpart of the quiz module, since it allows open-end questions. Thus the teacher can propose the students a stimulus (one or more questions, a problem, a text to be analyzed, …) and they can answer by submitting a file in any format, the fulfilling of an online form or in other ways requiring anyway the elaboration of a text or an hypertext. Both the task's text and the answer can include verbal text, symbolic expressions and figural representations.

The teacher can assess the product, communicate each student the outcomes of the assessment and require a further submission of an answer. In this activity, as in the other ones, it is possible to fix time restrictions to both the possibility to see the task's text and to submit the answer. It is also possible to subordinate reading and submission to the conclusion of other activities. These opportunities can be exploited to foster the students to be in contact with the course, for instance by tasks requiring to think about topics which are preliminaries with respect to the subjects of a lecture, with the constraint of doing it during the days immediately before it.

It is important to point out that the use of 'task' cannot be thought as the proposal of the classical mathematics exercise of a textbook. In fact in this case it should consist in requiring the student just a heaviness of something which does not give any added value to the paper task, on the contrary it requires the authoring effort of many mathematical symbols.

A sample of 'task' can be the following:

"*A friend of yours says that she cannot understand which a basis of a vector space is and ask you to explain it. Write what you say her in order to make her* *understand.*"

Such a request stake the multivariety, in the sense that it poses the task in the frame of peer communication so that the register is different from an official text, and anyway the written communication, that is diachronic, requires a certain accuracy in order to make oneself understand. Let us see an excerpt:

> "*Then, in order to make clear the concept of basis we need first of all to talk about vectors 'generators', linear dependence and independence of vectors in a generic vector space V […] it can be written as* $v_1h_1+v_2h_2+\ldots+v_nh_m$*, that is it can be defined as the sum of the products of the individual vectors with the related scalars*".

The beginning of the text seems to show care in 'make clear' the concept and thus it makes some preliminary remarks so that the following should be clear. Note that subsequently the same aim has been pursued staking a conversion in a different semiotic system: she goes from the symbolic language of the linear combination to the verbal language which explain the meaning of the written formula.

It is also true that the temptation of 'copying' the definition of basis from the textbook is great for many students. Anyway, the task generates at the same time a sort of 'competition' to wish to well done and sometimes it happens that someone 'copies too much', as in the following excerpt:

> "*It is defined basis B of a vector space V on the field K a set of vectors linearly independent and a set of generators of V.*
> ***Moreover B** is a basis if all the vectors in V are generated by B and the null vector is generated only by zero scalars*".

An analysis of the answers to the task is out of the scope of this paper. Anyway in the given sample, the teacher has set up a correspondence (dialogue) with the student starting from the adverb "Moreover" with the aim of investigating what was the meaning for the student and then to make come out that the two written sentences are equivalent.

More sophisticated tasks can ask the student to construct for instance a proof of a proposition organizing various given pieces and justifying the purpose of each piece, or, starting from the text of a proof, to answer questions requiring to make explicit the meaning or the purpose of some sentences, to convert them in formulas or to explain using verbal language (Albano & Ferrari, 2011).

The dialogic value of the module 'task' is in the opportunity that it offers to implement an iterative process of communication between tutor and student, consisting in a cycle such as: (i) goal-oriented online task, (ii) submission of a student's product, (iii) teachers' feedback and adjustment of the task, eventually requiring new submission, (iv) student's thinking generating new product, then the cycle can start again. The process *task – answer – assessment – new answer* - ... is apt not only to the progressive construction of the mathematical knowledge and procedures, but also to the thinking about the language and its progressive refinement.

Anyway a one-to-one communication model is not practical for large groups of students, as in the case of universities' classes. On the other hand it is needed to use a certain number of task to integrate the close-end quizzes in order to balance the limits highlighted in the previous section. Experimentations are in progress to investigate methods which foster the students in self-assessment processes, allowing them to think about their products and to realize their gaps without excessive interventions of the tutors (e.g. Albano, 2011). The experimentations are based on the following scheme. The teacher assigns a task. Each student, after (and only after) submitting her answer, can access a document containing a model of adequate answer, or a sequence of hints or facilities, which can depend on the nature of the task. These helping documents can be inserted into a *Lesson* (see next subsection) and followed by some questions requiring the student to assess her own product once she has looked at the answer's model or helping document, and to give a judgment on the usefulness of the received help. If the tool Lesson is not available, the documents and the self-assessment questions can be delivered by a *Shared Area*. These modalities of use of the module *Task*, which are to be defined and at the moment in experimentation, could allow from one hand to give the student the chance to assess her own product and to drive her in thinking about, on the other hand to manage a course even with not many human resources.

4.3 The module Lesson

The activity *Lesson* (in Moodle) allows to construct articulated paths where contents, in various formats (text, power point, videos, etc.) are delivered to students, with the possibility to insert check questions at the end. In case of getting through the check, the students are addressed towards subsequent contents, in case of failure they can be addressed towards recovery activities, even different according to the mistakes done, or suggested to study again the contents or to consult a glossary or other reference materials.

The lesson allows to plan integrated learning activities whose assessment can include various levels: pure and simple comprehension of texts as linguistic products, the contents' one, the procedures' one and the meta-cognitive awareness about ampler pieces. Among the disadvantages some limits common to quizzed and the difficulty to plan paths actually personalized should be included.

From the view point of the communication and of the languages, the lesson can incorporate a much ample range of representations, from verbal texts to the symbolic expressions, from the imagines to the videos. The possibilities of feedback for the students are more or less the same ones of the quizzes.

4.4 The module workshop

In the discursive approach to learning we have to take into account that the cognitive processes of talking, discussing and explaining to others (often in various modes) the concepts to be learnt foster a higher level and deeper thinking (Johnson & Johnson, 1987). Tools like the 'workshop' allow to implement time-restricted activities, based on role-plays, which engage the students as in the role of teacher and as in the role of student, which can foster such processes (Albano *et al.*, 2007). In the role of teacher a process of 'reciprocal peer questioning', that is 'problem posing', is put in action, which stakes various competencies and attitudes:

> «*I ought to better study the topics, because to be able to ask a question something much more complicated, almost more than to explain, because a question has to be well thought over, there is not method to formulate a question, there can be a method to answer, them it is something more difficult*»;
> «*to ask a non trivial question you have in any case to well know the topic, otherwise you cannot make a deep question*»;

and it modifies the attitudes too:

> «*they have helped me to put myself in the teacher's shoes, to see what he wants*».

Moreover it has been noted that such activities contribute to the construction of small virtual learning community. The one who participates to the working groups feels to be part of a 'community' which is in some sense *motivated to learn*. This awareness foster the starting up of meta-cognitive processes, of thinking about herself, about what has been learning, about the aims to be reached, which implies self-regulation processes and also include the search for help within the community (to address others in order to solve a problem or to make a concept clear):

> «*personally when I asked some questions, most of them were questions I was not able to answer, then I used it to understand what I could not understand* ».

Also this activity poses management problems: some students could not to stand the work pace, or could invest not much in problem posing, inventing only trivial problems. Moreover the problem of the tutors' role during the intermediate phases is posed, in particular whether they should or should not intervene when not well placed problems are posted. Finally, even here the question of the sustainability of such kind of activity with large groups of students.

4.5 The social tools: forum, wiki, blog

A platform, on one hand, allows to use not only an large range of semiotic systems, but a great variety of linguistic registers too. On the other hand, in most of the cases, this opportunity is not exploited since both the writer and the reader almost always adopt a minimalist use of the language, influenced by the idea that the essential thing is to reach the content 'apart from all'. Then it happens to come up against courses where the use of the language is monotonous (or always careless, or always elevated), whilst the students adopt more and more frequently superficial reading methods and, in some case, do not read at all.

The important thing is that the student has the chance to read and to produce texts correct and proper to the representation of the mathematical concepts, but also to use colloquial registers in those cases where they carry out an absolutely necessary cognitive function (in the communication, in the modeling and in the initial phases of all those mathematical activities). In order to understand the mathematical concepts it is no use so to master sophisticated registers as to be able to consciously go, both as transmitter and as receiver, from a register to another one. Activities such as *Forum*, *Wiki* and *Chat* are favored ground for using colloquial registers, even if often they are neglected by the students.

Anyway there is to say that here the semiotic constraint is much more posed, in the sense that these environments usually are thought exclusively for the communication in verbal language. Thus their use assumes to plan activities which can be 'restricted' to this only semiotic system. To avoid the editing difficulty, some platforms allow to enclose a file, but this solution weakens awfully the 'communicational' goal of such tools, because clearly it make all less readable and less 'collaborative'.

Conclusions

In our opinion research on e-learning in mathematics education requires the development of online courses with a sufficient amount of resources and activities, all of which should be accurately planned and designed. Of course this aim raises the problem of viability of a course like that (i.e. the amount of human resources needed to develop and to keep it).

When a course like that is available, plenty of research questions could be investigated. Some of them are require long-term studies. This is necessary when elearning is involved, if we want really to investigate its potential related to mathematical understanding rather than the transmission of mathematical contents. As examples, we mention some of the research questions that it would be worthwhile to address:

- Which resources or activities are most popular among students and why?
- How the less popular activities could be changed in order to increase their use, without losing effectiveness?
- Which resources or activities can best stimulate the development of mathematical reasoning?
- What are the outcomes of the use of resources an activities involving two semiotic systems or more?
- What are the outcomes of the use of resources an activities involving different varieties of verbal language?

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